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Higher Order Hulls in H^∞, II

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We give a complete characterization of the closed ideals in H^∞ with hull contained in the set G of nontrivial Gleason parts and show that these ideals are uniquely determined by their higher order hulls. We apply our theorem to finitely generated ideals and obtain an extension of results of J. Bourgain. We also determine generating sets for the class of all ideals with hull contained in G. © 2000 Academic Press

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Let H^∞ be the Banach algebra of all bounded analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. One way to study a Banach algebra is to understand the ideals in the algebra. In the case of H^∞, the maximal ideals have been well-studied. One can also look at closed ideals with certain special properties. For example, the closed prime ideals in H^∞ have been the object of much recent study. The goal, of course, is to understand the general closed ideals in the algebra and the hope is to use this to learn more about the algebra itself. In this paper, we will give a complete description of certain closed ideals. To do so, we first need to discuss our notation and a few definitions.

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It is well-known that the kernel of each nonzero multiplicative linear functional on H^∞ is a maximal ideal and that every maximal ideal can be thought of as the kernel of a nonzero complex multiplicative linear functional. For this reason, we call the space of nonzero complex multiplicative linear functionals on H^∞ the maximal ideal space of H^∞ and denote it by $M(H^\infty)$. When endowed with the weak-star topology, $M(H^\infty)$ becomes a compact Hausdorff space. Because H^∞ is a uniform algebra, we may identify a function $f \in H^\infty$ with its Gelfand transform, $\hat{f}$, defined by $\hat{f}(m) = m(f)$ for $m \in M(H^\infty)$. As usual, we identify $\mathbb{D}$ with a subset of $M(H^\infty)$. The Shilov boundary of H^∞ will be denoted by ∂H^∞. Let $f \in H^\infty$. The zero set of f in $M(H^\infty)$ is defined by $Z(f) = \{m \in M(H^\infty) : f(m) = 0\}$, whereas its zero set in $\mathbb{D}$ is given by $Z_{\mathbb{D}}(f) = \{z \in \mathbb{D} : f(z) = 0\}$. The hull, or zero set of an ideal I is the set $Z(I) = \bigcap_{f \in I} Z(f)$.

For two points x, m in $M(H^\infty)$, we define the pseudohyperbolic distance of x to m by

$$\rho(x, m) = \sup\{|f(m)|: f \in H^\infty, \|f\|_\infty \leqslant 1, f(x) = 0\}.$$

It is well-known that the relation defined on $M(H^\infty)$ by

$$x \sim m \Leftrightarrow \rho(x, m) < 1$$

defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point m is called the Gleason part of m and is denoted by $P(m)$. If the part, $P(m)$, consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. Hoffman's theory [9] shows that for every Gleason part $P(m)$ there is a continuous map L_m of $\mathbb{D}$ onto $P(m)$ with $L_m(0) = m$ such that $f \circ L_m$ is analytic on $\mathbb{D}$ for all $f \in H^\infty$. When the Gleason part of m is trivial, L_m is just a constant map. When $P(m)$ is nontrivial, the map L_m is a bijection. The set of all nontrivial points in $M(H^\infty)$ is denoted by G, and the set of all trivial points is denoted by Γ. Since $f \circ L_m \in H^\infty$, when $f(m) = 0$ it makes sense to talk about the order of the zero of f at m. For $m \in M(H^\infty)$ and $f \in H^\infty$ with $f(m) = 0$ we let

$$\operatorname{ord}(f, m) = \sup\{n \in \mathbb{N} : f = f_1 \cdots f_n, f_j \in H^\infty, f_j(m) = 0 \text{ for } j = 1, 2, ..., n\}.$$

If $f(m) \neq 0$, we say $\operatorname{ord}(f, m) = 0$. If I is an ideal in H^∞, we let $\operatorname{ord}(I, m) = \min\{\operatorname{ord}(f, m): f \in I\}$.

Hoffman showed that $\operatorname{ord}(f, m) = n$ if and only if $f \circ L_m$ has a zero of order n at 0. Moreover, if $\operatorname{ord}(f, m) = \infty$ for some $m \in M(H^\infty)$, then f

vanishes identically on the part $P(m)$ (see [9, pp. 79, 101]). If m is a trivial point, then $\operatorname{ord}(f, m) = \infty$, whenever $f(m) = 0$.

While it is relatively easy to determine the order of the zero of f at m, it is very difficult to determine when f belongs to a given closed ideal. In this paper we will obtain the following result:

THEOREM. *Let I be a closed ideal in H^∞ satisfying $Z(I) \subseteq G$. Then*

$$I = \{f \in H^\infty : \operatorname{ord}(f, x) \geqslant \operatorname{ord}(I, x)\ \forall x \in Z(I)\}.$$

In particular, I is divisorial; that is, I is an intersection of primary ideals.

While the assumption that $Z(I) \subseteq G$ may look as though it can be omitted, an example due to Bourgain shows that it is indeed necessary. In fact, Bourgain [2] (see also [6]) showed that there exist two Blaschke products B and C such that BC does not belong to the closure of the ideal I generated by B^2 and C^2. Note that

$$\operatorname{ord}(BC, m) \geqslant 2 \min\{\operatorname{ord}(B, m), \operatorname{ord}(C, m)\} = \operatorname{ord}(I, m).$$

The reason for this phenomenon is that, in his example, $\operatorname{ord}(I, m)$ is infinite for some $m \in M(H^\infty)$. Thus, the hull of this ideal necessarily meets the set of trivial points.

We are now able to define the objects of primary interest in this paper, the higher order zero sets. Let $\mathbb{N} = \{1, 2, ...\}$. For $n \in \mathbb{N} \cup \{\infty\}$ we define

$$E_n(f) = \{m \in M(H^\infty) : \operatorname{ord}(f, m) \geqslant n\}.$$

Thus $E_1(f)$ is the zero set of f. Given an ideal I and an integer n, one can also define the higher order zero sets or hulls of ideals by $E_n(I) = \bigcap_{f \in I} E_n(f)$; that is, we let

$$E_n(I) = \{x \in M(H^\infty) : \operatorname{ord}(f, x) \geqslant n \text{ for every } f \in I\}.$$

Note that $E_1(I)$ is the hull of I. Finally, given an ideal I in H^∞, we let

$$\begin{aligned} &I[E_1(I), ..., E_N(I)] \\ &\qquad = \{f \in H^\infty : \operatorname{ord}(f, x) \geqslant n \text{ for every } x \in E_n(I), n = 1, 2, ..., N\}. \end{aligned}$$

Our main result in this paper shows that if I is a closed ideal such that $Z(I) \subseteq G$, then there exists a positive integer N such that $E_n(I) = \varnothing$ for $n \geqslant N + 1$ and

$$I = I[E_1(I), E_2(I), ..., E_N(I)].$$

Using an idea of Tolokonnikov on the relationship between the higher order hulls and higher order pseudohyperbolic derivatives $D^{(j)}$, we obtain yet another representation of these ideals above:

$$I = \{ f \in H^\infty : D^{(j-1)} f = 0 \text{ on } E_j(I) \text{ for all } j = 1, 2, ..., N \}.$$

This result may be viewed as a Malgrange–Tougeron type theorem. Recall that an ideal I in the Banach algebra $A = C^n([0,1])$ of all functions with continuous derivatives up to the order n on the compact interval $[0,1]$ is closed if and only if there exist closed subsets E_j, $(j = 0, 1, ..., n)$ of $[0,1]$, $E_n \subseteq E_{n-1} \subseteq \cdots \subseteq E_1 \subseteq E_0$ and $E_j \backslash E_{j+1}$ being discrete sets for $j = 0, 1, ..., n-1$, such that

$$I = \{ f \in A : \forall j \in \{0, 1, ..., n\}, f^{(j)} = 0 \text{ on } E_j \}.$$

A similar result also holds for the algebra $A^n(\mathbb{D})$ of all analytic functions in the open unit disk whose nth derivative extend continuously to $\overline{\mathbb{D}}$. This is a result of B. Korenblum [12]. Let us point out, however, that these algebras are *not* uniform algebras (for $n \geqslant 1$).

In Section 3 we shall study generating sets for the class $\mathcal{G}$ of all ideals with hull contained in the set of nontrivial Gleason parts. It is well known [5, 17] that these ideals are generated by a set of Carleson–Newman Blaschke products. It will be shown that one can control the degree of these Blaschke products as well as the localization of their zeros. Thus we obtain new results concerning the fine structure of that class of ideals.

We conclude this paper in Section 4, where we apply the results in Section 2 to finitely generated ideals in H^∞ and extend, for our class $\mathcal{G}$, a result of Bourgain on closures of finitely generated ideals. We also study finite products of ideals.

The results of this paper were proven earlier by P. Gorkin and R. Mortini using a longer and rather technical, but related approach (see [GoMo]). That manuscript can be viewed at the third author's website http://poncelet.sciences.univ-metz.fr/~mortini.

1. NOTATION AND PRELIMINARIES

We begin by recalling useful definitions and theorems that we will need in this paper. Then we prove some lemmas.

In the theory of bounded analytic functions, one of the most useful kinds of functions are interpolating Blaschke products. Recall that a sequence $(a_n)_{n \in \mathbb{N}}$ and the associated Blaschke product $B(z) = \prod_{n=1}^{\infty} (\bar{a}_n / |a_n|)(a_n - z)/(1 - \bar{a}_n z)$ are called interpolating if for every bounded sequence $(w_n)_{n \in \mathbb{N}}$

there exists a function $f \in H^\infty$ such that $f(a_n) = w_n$ for all $n \in \mathbb{N}$. We will also consider finite Blaschke products with distinct zeros as interpolating Blaschke products. By Carleson's theorem on interpolation (see [4, p. 287]) we know that (a_n) is interpolating if and only if

$$(C) \qquad \delta(\{a_n\}_n) := \delta(B) := \inf_{k \in \mathbb{N}} \prod_{j:\, j \neq k} \left| \frac{a_j - a_k}{1 - \bar{a}_j a_k} \right| \geqslant \delta > 0.$$

The constant $\delta(B)$ is called the uniform separating constant of (a_n) or B. The associated interpolation constant K is defined by

$$K = \sup_{\|w_n\|_\infty \leqslant 1} \inf\{\|f\|_\infty : f(a_n) = w_n, \text{ for all } n \in \mathbb{N},\, f \in H^\infty\}.$$

It is well-known (see [4, p. 287]) that $\frac{1}{\delta(B)} \leqslant K \leqslant \frac{c}{\delta}(1 + \log \frac{1}{\delta})$ for some universal constant c.

A result of Hoffman [9] stating that a point m in the maximal ideal space is nontrivial if and only if m lies in the closure of an interpolating sequence in $\mathbb{D}$ is one key fact that we will use.

A Blaschke product B that equals a finite product of interpolating Blaschke products is also called a Carleson–Newman Blaschke product. Such a Blaschke product is said to have order p if it can be written as a product of p interpolating Blaschke products, but not written as a product of $p - 1$ or fewer interpolating Blaschke products. A Carleson–Newman Blaschke product $\prod_{j=1}^{p} b_j$ is interpolating if and only if the zero sets $Z(b_j)$ are pairwise disjoint.

We recall here a useful result due to K. Hoffman about the constants associated with interpolating Blaschke products. Recall that for $S \subseteq M(H^\infty)$ and $m \in M(H^\infty)$, the distance of m to S is given by $\rho(m, S) = \inf\{\rho(m, s) : s \in S\}$.

Hoffman's Lemma 1.0 ([9, p. 86, 106] and [4, p. 404, 310]). *Let* $0 < \delta < 1$, $0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta$, *(that is,* $0 < \eta < \rho(\delta, \eta)$*) and let*

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

Furthermore, if B is any interpolating Blaschke product with zeros $\{z_n\}$ such that

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2)\, |B'(z_n)| \geqslant \delta,$$

then

1. $Z(B)$ *is the closure of the zero set of* B *in* $\mathbb{D}$,
2. $\rho(x, y) \geqslant \delta$ *for any* $x, y \in Z(B)$, $x \neq y$, *and*
3. $\{m \in M(H^\infty) : |B(m)| < \varepsilon\} \subseteq \{m \in M(H^\infty) : \rho(m, Z(B)) < \eta\}$
$\subseteq \{m \in M(H^\infty) : |B(m)| < \eta\}$.

Moreover, the collection of closures of the pseudohyperbolic disks

$$D(m, \eta) = \{x \in M(H^\infty) : \rho(m, x) < \eta\}$$

for $m \in Z(B)$ *are pairwise disjoint.*

4. *Let* $\{z_n\}$ *be an interpolating sequence with* $\delta(\{z_n\}) = \delta$, *and let* $\rho(w_n, z_n) < \eta$. *Then* $\{w_n\}$ *is an interpolating sequence with*

$$\delta(\{w_n\}) > \frac{\delta - \dfrac{2\eta}{1+\eta^2}}{1 - \delta \dfrac{2\eta}{1+\eta^2}}.$$

We note also that $(1 - \sqrt{1-\delta^2})/\delta$ is a monotonically increasing function of $\delta \in (0, 1)$, that $\varepsilon < \eta < \delta$ and that $0 < (1 - \sqrt{1-\delta^2})/\delta < \delta$. Moreover, it is easy to see that $\eta < 2\eta/(1+\eta^2) < \delta$ is equivalent to $0 < \eta < \rho(\delta, \eta)$.

With the exception of (2), these results are stated in Hoffman's paper. Although (2) is not explicitly stated, one can prove it as follows.

Let $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ be disjoint subsets of the zeros of B in $\mathbb{D}$ such that $x \in \overline{\{u_n\}}$ and $y \in \overline{\{v_n\}}$. Let B_1 denote the subproduct of B associated with $\{u_n\}$, and let B_2 denote the subproduct of B associated with $\{v_n\}$. Then, for each n, we see that

$$|B_2(u_n)| \geqslant \prod_{k:\, z_k \neq u_n} \rho(z_k, u_n) \geqslant \delta.$$

Since B_2 is continuous on the maximal ideal space, $|B_2(x)| \geqslant \delta$ and $B_2(y) = 0$. Thus

$$\rho(x, y) = \sup\{|f(x)| : f \in H^\infty, \|f\|_\infty \leqslant 1, f(y) = 0\} \geqslant |B_2(x)| \geqslant \delta.$$

The following easy lemma will be used several times throughout this paper.

Lemma 1.1. *Let* I *be an ideal in* H^∞ *and suppose that* $f_j = C_j h_j \in I$, $j = 1, ..., n$, *for some* $C_j, h_j \in H^\infty$ *satisfying*

$$\bigcap_{1 \leqslant j \leqslant n} Z(h_j) \cap Z(I) = \varnothing.$$

Then $\prod_{j=1}^n C_j \in I$.

Proof. By our hypothesis the ideal generated by I and the h_j, $j=1, ..., n$ equals H^∞. Hence there exists a function $f \in I$ and functions $g_j \in H^\infty$ such that $1 = \sum_{j=1}^n g_j h_j + f$. Therefore

$$\prod_{j=1}^n C_j = \sum_{j=1}^n \left(\prod_{k \neq j} C_k \right) g_j f_j + \left(\prod_{j=1}^n C_j \right) f \in I + I \subseteq I. \quad \blacksquare$$

LEMMA 1.2. *Let I be an ideal in H^∞. Then the function ω: $M(H^\infty) \to \mathbb{N}_0 \cup \{\infty\}$, defined by $\omega(x) = \operatorname{ord}(I, x)$, is upper semicontinuous.*

Proof. Using the upper semicontinuity of the function $x \to \operatorname{ord}(f, x)$, (see [5, p. 152]), and the fact that for any real number α the equality

$$\{x\colon \operatorname{ord}(I, x) \geqslant \alpha\} = \bigcap_{f \in I} \{x\colon \operatorname{ord}(f, x) \geqslant \alpha\}$$

holds, we immediately obtain that these sets are closed, which yields the assertion. $\blacksquare$

LEMMA 1.3. *If I is an ideal in H^∞ satisfying $Z(I) \subseteq G$, then*

$$N = \sup_x \operatorname{ord}(I, x) < \infty.$$

Proof. Assuming the contrary, there exists a sequence (x_n) in G such that $\operatorname{ord}(I, x_n) \to \infty$. Let x be a cluster point of the x_n. Then $x \in Z(I)$. By Lemma 1.2, the upper semicontinuity of $\operatorname{ord}(I, x)$ yields that $\operatorname{ord}(I, x) = \infty$. Thus, either x is a trivial point, that is $x \in \Gamma$, or every function $f \in I$ vanishes identically on the closure of the part $P(x)$ (see [4, p. 403]). By Budde ([3, p. 370]), $\overline{P(x)}$ contains a trivial point m. Hence in both cases, x or m is in $\Gamma \cap Z(I)$, a contradiction. $\blacksquare$

The following lemma surely is known. Since we could not find a reference for this topological fact, we present for the readers convenience a short elementary proof.

LEMMA 1.4. *The finite union of compact, totally disconnected subspaces of a Hausdorff space is totally disconnected.*

Proof. Let S_1 and S_2 be two totally disconnected compact subspaces of a Hausdorff space X. Note that the clopen ($=$open and closed) sets form a topological basis for every totally disconnected, compact Hausdorff space. Let x and y be two different points in $S_1 \cup S_2$. If $x \in S_1 \backslash S_2$, then we choose a clopen set $C \subseteq S_1$ with $x \in C \subseteq S_1 \backslash (S_1 \cap S_2)$, but $y \notin C$. Since

$(S_1 \cup S_2)\backslash C = (S_1 \backslash C) \cup S_2$ is compact, we see that C is clopen in $S_1 \cup S_2$, too. Hence x and y do not belong to the same connected components in $S_1 \cup S_2$.

If $x, y \in S_1 \cap S_2$, then choose a clopen set $C \subseteq S_1$ such that $x \in C$ and $y \in C^* := S_1 \backslash C$. Now take a clopen set $K \subseteq S_2$ such that $C \cap S_2 \subseteq K$ and $C^* \cap S_2 \subseteq K^* := S_2 \backslash K$. Since $C^* \cap K = \emptyset$, there exists $U \subseteq X$ open so that $U \cap S_2 = K$ and $U \cap C^* = \emptyset$. We may assume that $C \subseteq U$, for otherwise replace U by $U \cup (S_2 \cup C^*)^c$. Thus

$$U \cap (S_1 \cup S_2) = (U \cap S_1) \cup (U \cap S_2) = (C \cap S_1) \cup (K \cap S_2).$$

Therefore the latter set is clopen in $S_1 \cup S_2$. It contains x, but not y. Again, x and y do not belong to the same connected components in $S_1 \cup S_2$. ∎

PROPOSITION 1.5. *The zero set of a Carleson–Newman Blaschke product is totally disconnected.*

Proof. This follows from the fact that the zero set of an interpolating Blaschke product is homeomorphic to the Stone–Cech compactification of the integers—hence totally disconnected—and Lemma 1.4. ∎

2. CLOSED IDEALS WITH HULL IN G

We shall now present the main result of this paper: a complete characterization of the closed ideals in H^∞ with hull contained in the set of nontrivial Gleason parts. First we note that by [17] or [5, p. 153] *every* ideal in H^∞ whose hull does not intersect the set of trivial points, contains a Carleson–Newman Blaschke product. But actually more holds. In fact each ideal $I \subseteq H^\infty$ with $Z(I) \subseteq G$ is generated (algebraically) by Carleson–Newman Blaschke products. This is a consequence of the fact that whenever B is a Carleson–Newman Blaschke product contained in I, then for every $f \in I$ the function $g_f = B + \lambda f$ is a Carleson–Newman Blaschke product times an invertible outer function, whenever $\lambda > 0$ is small enough (see [8]). Thus, the ideals we are concerned with, are generated by Carleson–Newman Blaschke products.

LEMMA 2.1. *Let B_j, $j = 1, ..., n$, be interpolating Blaschke products vanishing at $x \in M(H^\infty)$ and let U be a neighborhood of x with $U \subseteq \bigcap_{1 \leqslant j \leqslant n} \{|B_j| < \varepsilon\}$. Then for every small $\gamma > 0$, there exists an open set V with $x \in V \subseteq \overline{V} \subseteq U$ such that $V \cap \mathbb{D}$ has the form*

$$V \cap \mathbb{D} = \bigcup_{1 \leqslant j \leqslant n} \bigcup_{v \in \mathbb{N}} D(a_{v,j}, \gamma),$$

where $\{a_{\nu,j}\colon \nu\in\mathbb{N}\}$ *is a subset of the zeros of* B_j *in* $\mathbb{D}$ *satisfying* $\rho(a_{\nu,j}, a_{\nu,k}) < \gamma$ *for* $1 \leqslant j, k \leqslant n$ *and* $\nu \in \mathbb{N}$.

Proof. Choose an interpolating Blaschke product b with $b(x)=0$ such that $\{|b|<\varepsilon'\}\subseteq U$ for some $\varepsilon'<\varepsilon$. Let $(z_{\nu(\alpha)})_\alpha$ be a subnet of the zero sequence of b converging to x. Choose γ^* and γ so that $2\gamma^*/(1+(\gamma^*)^2)<\gamma$ and $2\gamma/(1+\gamma^2)<\varepsilon'$. By [10, p. 551] there exist subnets (indexed by the same directed set) $(a_{\nu(\alpha),j})_\alpha$ of the zero sequences of B_j such that for all α sufficiently "large," say $\alpha\geqslant\alpha_0$, we have $\gamma^*\geqslant\rho(z_{\nu(\alpha)}, a_{\nu(\alpha),j})\to 0$. Then it is easy to check that for

$$\Omega=\bigcup_{\alpha\geqslant\alpha_0}\bigcup_{j=1}^{n} D(a_{\nu(\alpha),j},\gamma)$$

its closure $\bar{\Omega}$ in $M(H^\infty)$ satisfies $\bar{\Omega}^\circ\subseteq\{|b|<\varepsilon'\}\subseteq U$. The set $V=\bar{\Omega}^\circ$ now does the job. ∎

Theorem 2.2. *Let I be a closed ideal in H^∞ satisfying $Z(I)\subseteq G$. Then*

$$I=\{f\in H^\infty : \operatorname{ord}(f,x)\geqslant\operatorname{ord}(I,x)\ \forall x\in Z(I)\}.$$

In particular, I is an intersection of primary ideals.

Proof. Let

$$J=\{f\in H^\infty : \operatorname{ord}(f,x)\geqslant\operatorname{ord}(I,x)\ \forall x\in Z(I)\}.$$

Claim 1. *J is a closed ideal and $I\subseteq J$. Moreover, $Z(J)=Z(I)$.*

Proof. Obviously J is an ideal containing I such that $Z(J)=Z(I)$. To prove the closedness, let $f_n\in J$ converge uniformly to $f\in H^\infty$. Choose $x\in Z(I)$ and let L_x be the Hoffman map associated with x. Then $f_n\circ L_x$ converges uniformly to $f\circ L_x$. Since those functions are analytic, we see that $\operatorname{ord}(f,x)=\operatorname{ord}(f\circ L_x,0)\geqslant\operatorname{ord}(I,x)$. Hence $f\in J$.

Claim 2. *$Z(I)$ is totally disconnected.*

Proof. Since $Z(I)\subseteq G$, the ideal I contains by [17] or [5, p. 153], a Carleson–Newman Blaschke product B. Therefore, $Z(I)$ is contained in the zero set $Z(B)$ of B. Since the total disconnectivity is inherited by subspaces, by Proposition 1.5 we obtain that $Z(I)$ is totally disconnected, too.

As noted above, since $Z(I)$ and $Z(J)$ are contained in G, they are generated by Carleson–Newman Blaschke products. Hence, in order to prove that $I = J$, it is sufficient to show that whenever Φ is a Carleson–Newman Blaschke product contained in J, then $\Phi \in I$.

So let $\Phi \in J$ be a Carleson–Newman Blaschke product and write $\Phi = B_1 B_2 \cdots B_k$, where B_j is an interpolating Blaschke product. Take $\delta > 0$ such that $\delta(B_j) \geqslant \delta$ for $j \in \{1, ..., k\}$ and $\eta > 0$ so that $0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta$ and

$$\frac{\delta - \dfrac{2\eta}{1+\eta^2}}{1 - \delta \dfrac{2\eta}{1+\eta^2}}.$$

Let ε be sufficiently small, say $0 < \varepsilon < \eta \, \frac{\delta - \eta}{1 - \delta\eta}$.

Claim 3. *There exists $\sigma > 0$—depending only on δ—, such that for all ε above, there exist interpolating Blaschke products $b_1, b_2, ..., b_k$ such that*

$$\Psi := \prod_{j=1}^{k} b_j \in I \tag{1}$$

$$Z(b_j) \subseteq \{ |B_j| < \varepsilon \} \qquad (j = 1, ..., k) \tag{2}$$

$$\delta(b_j) > \sigma \qquad \textit{for} \quad j \in \{1, ..., k\}. \tag{3}$$

Proof. Let $x \in Z(I)$. Choose $0 < \eta_x < \varepsilon$ so that

$$\eta_x < \min_{\nu:\, B_\nu(x) \neq 0} |B_\nu(x)|.$$

Let $0 < \varepsilon_x < \eta_x((\delta - \eta_x)/(1 - \delta\eta_x))$. For $x \in Z(I)$, take an open neighborhood U_x of x in $M(H^\infty)$ such that

$$U_x \subseteq \bigcap_{\nu:\, B_\nu(x) = 0} \{ |B_\nu| < \varepsilon_x \} \cap \bigcap_{\nu:\, B_\nu(x) \neq 0} \{ |B_\nu| > \eta_x \}. \tag{4}$$

Choose $f_x \in I$ satisfying $\operatorname{ord}(f_x, x) = \operatorname{ord}(I, x)$. Let $s = s(x) = \operatorname{ord}(f_x, x)$. By Hoffman's factorization theorem [9, p. 100], $f_x = c_{x,1} c_{x,2} \cdots c_{x,s} \, g_x$, where $c_{x,j}$ is an interpolating Blaschke product with $c_{x,j}(x) = 0$ and $g_x \in H^\infty$, $g_x(x) \neq 0$. We may assume without loss of generality that $\delta(c_{x,j}) > \delta$.

By Claim 2, $Z(I)$ is totally disconnected. Thus there exist open sets V_x in $M(H^\infty)$ such that

$$x \in V_x \subseteq \overline{V_x} \subseteq U_x, \qquad V_x \cap Z(g_x) = \varnothing, \tag{5}$$

and $V_x \cap Z(I)$ is a clopen subset of $Z(I)$. Let $\gamma > 0$ satisfy $2\gamma/(1 + \gamma^2) < \varepsilon$.

By Lemma 2.1, we may assume that $V_x \cap \mathbb{D}$ has the form

$$V_x \cap \mathbb{D} = \bigcup_{j:\, B_j(x) = 0} \ \bigcup_{\nu \in \mathbb{N}} D(a_{\nu, j}^{(x)}, \gamma_x),$$

where $\gamma_x < \min\{\varepsilon_x, \gamma\}$, $\{a_{\nu,j}^{(x)} : \nu \in \mathbb{N}\} \subseteq Z_{\mathbb{D}}(B_j)$ and $\rho(a_{\nu,j}^{(x)}, a_{\nu,i}^{(x)}) < \gamma_x$ whenever $B_j(x) = B_i(x) = 0$.

Call the union $\bigcup_{j:\, B_j(x)=0} D(a^{(x)}_{\nu,\, j}, \gamma_x)$ the building blocks of V_x.

Since $Z(I)$ is compact, there exist $x_1, x_2, ..., x_n \in Z(I)$ such that

$$Z(I) \subseteq \bigcup_{j=1}^{n} V_{x_j} \qquad \text{and} \qquad V_{x_j} \cap Z(I) \not\subseteq \bigcup_{i:\, i \neq j} (V_{x_i} \cap Z(I)).$$

We shall now modify the V_{x_j} a little bit, in order to guarantee that building blocks comming from x_j do not intersect the building blocks comming from x_i, whenever x_j and x_i are zeros of a common Blaschke product B_ν.

Let $1 \leqslant i_1 < i_2 \leqslant n$, $I_1 = \{ j \in \{1, ..., k\} : B_j(x_{i_1}) = 0\}$ and $I_2 = \{ j \in \{1, ..., k\} : B_j(x_{i_2}) = 0\}$. Suppose that $\varepsilon_{x_{i_1}} \leqslant \varepsilon_{x_{i_2}}$, these constants being those chosen above (see (4)). Assume that $I_1 \cap I_2 \neq \emptyset$; (we are not interested in the other case). Note that the building blocks for $V_{x_{i_r}}$ consists of the same number of disks as there are elements in I_r, $(r = 1, 2)$. Then the choice (4) of U_x guarantees that a given building block of $V_{x_{i_1}}$ is either disjoint from (those of) $V_{x_{i_2}}$ or that it is entirely contained in a single building block of $V_{x_{i_2}}$. Hence $I_1 \subseteq I_2$.

Now we delete from $V_{x_{i_1}}$ all the building blocks entirely contained in $V_{x_{i_2}}$, i.e. those which are centered at the same group of zeros as those from $V_{x_{i_2}}$. Do this for every pair (i_1, i_2) of different indices out of $\{1, ..., n\}$. On the whole, this does not change the union $\bigcup_{j=1}^{n} V_{x_j}$. Let us denote the modified V_{x_j} by Ω_j. Recalling that for $\mathcal{O}$ open in $\mathbb{D}$ one has $\mathcal{O} \subseteq \overline{\mathcal{O}}^\circ$, we see that $\overline{\Omega_j}^\circ \subseteq U_{x_j}$ and $Z(I) \subseteq \bigcup \overline{\Omega_j}^\circ$.

Now choose open subsets $\mathcal{O}_j$ of the $\overline{\Omega_j}^\circ$ such that $Z(I) \cap \mathcal{O}_j$ is clopen in $Z(I)$ and so that $Z(I) \subseteq \bigcup \mathcal{O}_j$.

Let $A_1 = \mathcal{O}_1 \cap Z(I)$ and $A_j = (\mathcal{O}_j \cap Z(I)) \setminus \bigcup_{i=1}^{j-1} (\mathcal{O}_i \cap Z(I))$, $j = 2, ..., n$. Then the A_j are pairwise disjoint, nonvoid clopen subsets of $Z(I)$ such that $\bigcup_{j=1}^{n} A_j = Z(I)$ and $A_j \subseteq \mathcal{O}_j \cap Z(I)$.

Take an open subset W_j in $M(H^\infty)$ such that

$$A_j \subseteq W_j \subseteq \overline{W_j} \subseteq \mathcal{O}_j, \qquad \overline{W_j} \cap \overline{W_i} = \emptyset \quad \text{if} \quad i \neq j, \qquad A_j = W_j \cap Z(I).$$

Then we see that $W_j \cap Z(I) = \overline{W_j} \cap Z(I)$ and

$$Z(I) \subseteq \bigcup_{j=1}^{n} W_j. \tag{6}$$

We note that x_j does not necessarily belong to W_j.

Let $C_{x_j,\, i}$ be the subproduct of $c_{x_j,\, i}$ with $Z(C_{x_j,\, i}) \cap \mathbb{D} = Z(c_{x_j,\, i}) \cap \mathbb{D} \cap W_j$.

Then $Z(C_{x_j, i}) \subseteq \overline{W}_j$. Moreover,

$$f_{x_j} = C_j h_j, \qquad \text{where} \quad C_j = \prod_{i=1}^{s(x_j)} C_{x_j, i} \qquad \text{and} \qquad h_j \in H^\infty, \qquad (j = 1, ..., n).$$

By (5) we have $Z(h_j) \cap W_j = \varnothing$.

Hence $\operatorname{ord}(C_j, x) \geqslant \operatorname{ord}(I, x)$ for every $x \in W_j \cap Z(I)$. By (6) we obtain

$$\bigcap_{j=1}^{n} Z(h_j) \cap Z(I) = \varnothing. \tag{7}$$

By Lemma 1.1 we have that

$$\prod_{j=1}^{n} C_j \in I. \tag{8}$$

We shall now order the zeros of the Carleson–Newman Blaschke products C_j such that $\prod_{j=1}^{n} C_j = \prod_{p=1}^{k} b_p$, where the b_p are interpolating Blaschke products satisfying the assertions (1)–(3).

Recall that $\Phi = \prod_{j=1}^{k} B_j$. Since $\operatorname{ord}(\Phi, x_j) \geqslant s(x_j)$ for $1 \leqslant j \leqslant n$, there exists $1 \leqslant \ell_1(j) < \ell_2(j) < \cdots < \ell_{s(x_j)}(j) \leqslant k$ such that

$$B_{\ell_1(j)}(x_j) = B_{\ell_2(j)}(x_j) = \cdots = B_{\ell_{s(x_j)}(j)}(x_j) = 0, \qquad j = 1, ..., n. \tag{9}$$

Now let us construct the functions b_p.

Fix j. Associate with $C_{x_j, i}$, $1 \leqslant i \leqslant s(x_j)$, the ith function (not necessarily the ith coordinate) of the k-tuple $(B_1, B_2, ..., B_k)$ vanishing at x_j. Note that the number of functions B_v which vanish at x_j is at least as big as $s(x_j)$. For $p \in \{1, ..., k\}$ define the Blaschke product b_p as the product of all the $C_{x_j, i}$ which are associated with B_p (if there is no such $C_{x_j, i}$, then we let $b_p = 1$; note that for fixed j there is at most one v for which $\ell_v(j) = p$).

In other words

$$b_p = \prod_{(j, v):\, \ell_v(j) = p} C_{x_j, v}.$$

By our construction

$$Z(b_p) \subseteq \bigcup_{j:\, B_p(x_j) = 0} \overline{W}_j \subseteq \bigcup_{j:\, B_p(x_j) = 0} \mathcal{O}_j \subseteq \bigcup_{j:\, B_p(x_j) = 0} U_{x_j} \overset{(4)}{\subseteq} \{|B_p| < \varepsilon\}; \tag{10}$$

hence (2) is satisfied. Since $\prod_{p=1}^{k} b_p = \prod_{j=1}^{n} C_j$, we get from (8) that $\prod_{p=1}^{k} b_p \in I$.

Because the zero sets $Z(C_{x_j,\nu})$ of the interpolating Blaschke products $C_{x_j,\nu}$ are contained in the pairwise disjoint sets $\overline{W_j}$, we obtain that b_p is an interpolating Blaschke product. Let $\sigma = \min_{1 \leqslant j \leqslant k} \delta(b_j)$. We show that $\sigma \geqslant \frac{\delta}{2}$.

Recall that the zeros of the $C_{x_j,\nu}$ come from those of $c_{x_j,\nu}$ and are contained in $\bigcap_{\nu:\, B_\nu(x_j)=0} \{|B_\nu| < \varepsilon\}$. Moreover, each $D(a,\eta)$, $a \in \mathbb{D}$, contains at most one zero of $c_{x_j,\nu}$. Fix p. Now use

(a) $Z_{\mathbb{D}}(C_{x_j,\nu}) \subseteq W_j \cap \mathbb{D}$,

(b) $W_j \cap \mathbb{D} \subseteq \Omega_j$,

and, most important, that whenever $B_p(x_j) = B_p(x_k) = 0$, then $D(a^{(x_j)}_{\nu,p}, \gamma_{x_j}) \cap D(a^{(x_k)}_{\mu,p}, \gamma_{x_k}) = \emptyset$ for every ν, μ. Note that $\gamma_x \leqslant \gamma < \eta$. By Hoffman's Lemma we then have $D(a^{(x_j)}_{\nu,p}, \eta) \cap D(a^{(x_k)}_{\mu,p}, \eta) = \emptyset$.

Hence, by (10), we see that the zeros in $\mathbb{D}$ of b_p are contained in $\bigcup D(a_n, \eta)$, where a_n runs through all the zeros of B_p in $\mathbb{D}$ and that each of these disks contains at most one of the zeros of b_p. Hence, by the fourth assertion of Hoffman's lemma, we see that $\delta(b_p) \geqslant \frac{\delta}{2}$. Thus (1)–(3) are fulfilled.

We are now ready to complete the proof of the theorem.

Claim 4. *There exists a constant κ depending only on δ such that*

$$\operatorname{dist}(\Phi, \Psi H^\infty) \leqslant \kappa\varepsilon,$$

where Φ is the given Carleson–Newman Blaschke product in J and $\Psi \in I$ is the Carleson–Newman Blaschke product constructed in Claim 3.

Proof. Recall that $\Phi = \prod_{j=1}^k B_j$ and that $\Psi = \prod_{j=1}^k b_j$, where $Z(b_j) \subseteq \{|B_j| < \varepsilon\}$. Consider the set of all solutions h_j to the interpolation problems

$$h_j|_{Z(b_j) \cap \mathbb{D}} = B_j|_{Z(b_j) \cap \mathbb{D}}.$$

Note that $\delta(b_j) > \sigma := \delta/2$. Let $K(\sigma)$ be the interpolation constant associated with σ. Then there exist solutions h_j satisfying $\|h_j\| \leqslant K(\sigma) \sup_{Z(b_j) \cap \mathbb{D}} |B_j|$.

Since $Z(b_j) \subseteq \{|B_j| < \varepsilon\}$, we obtain solutions satisfying $\|h_j\| \leqslant \varepsilon K(\sigma)$. Moreover, $B_j = h_j + g_j b_j$ for some $g_j \in H^\infty$. Now we use the following elementary inequality:

$$\left| \prod_{j=1}^k a_j - \prod_{j=1}^k b_j \right| \leqslant M^k \sum_{j=1}^k |a_j - b_j|,$$

$$\text{where} \quad a_j, b_j \in \mathbb{C}, \qquad |a_j|, |b_j| \leqslant M.$$

Together with the estimate $\|b_j\, g_j\| \leqslant 1+\varepsilon K(\sigma)$ we obtain:

$$\left\| \prod_{j=1}^{k} B_j - \prod_{j=1}^{k} b_j\, g_j \right\| \leqslant (1+\varepsilon K(\sigma))^k \sum_{j=1}^{k} \|B_j - b_j\, g_j\|$$

$$= (1+\varepsilon K(\sigma))^k \sum_{j=1}^{k} \|h_j\| \leqslant (1+K(\sigma))^k\, kK(\sigma)\varepsilon.$$

Let $\kappa = kK(\sigma)(1+K(\sigma))^k$. Hence $\operatorname{dist}(\Phi, \Psi H^\infty) \leqslant \kappa\varepsilon$.

By letting $\varepsilon \to 0$, from claim (4) we now get that $\operatorname{dist}(\Phi, I)=0$. Hence $\Phi \in \overline{I} = I$. ∎

The notion of higher order hulls now gives us insight into the finer structure of this class of ideals. First we note that for every $n \in \mathbb{N} \cup \{\infty\}$ the sets $E_n(f) = \{x \in M(H^\infty) : \operatorname{ord}(f, x) \geqslant n\}$ are closed, or, which is equivalent, the function $\operatorname{ord}(f, \cdot)$ is upper semicontinuous (see [5, p. 152]). Moreover, we have the following relations to pseudohyperbolic derivatives:

Define as in [17], for $f \in H^\infty$ the pseudohyperbolic derivatives by

$$D^j f(z) = \frac{d^j}{d\zeta^j} f\left(\frac{z+\zeta}{1+\bar{z}\zeta}\right)\bigg|_{\zeta=0}$$

for any positive integer j. Thus $D^1 f(z) = (1-|z|^2)\, f'(z)$. We let $D^0(f)=f$. Using the Cauchy integral formula for functions in the unit ball of H^∞, it is easy to see that $|D^j f| \leqslant j!$ for all $j \in \mathbb{N}$. Moreover, we have [9, p. 94]

$$D^j f(z) = \sum_{k=1}^{j} a_k \bar{z}^{j-k} (1-|z|^2)^k f^{(k)}(z)$$

for some constants a_k. By Hoffman's theory (see [9, p. 93]), for each j the functions $D^j f$ extend continuously to the whole spectrum $M(H^\infty)$ of H^∞. These extensions will also be denoted $D^j f$.

The following result is implicitely in Hoffman [9, p. 101]. We cite it for better references. Readers not familiar with H^∞-theory will find a detailed but short proof in [GoMo, Lemma 1.1]

LEMMA 2.3. *Let $f \in H^\infty$. Then for $n \in \mathbb{N}$ we have*

$$E_n(f) = \left\{ m \in M(H^\infty) : \sum_{k=0}^{n-1} |D^k f(m)| = 0 \right\},$$

and

$$E_\infty(f) = \{m \in M(H^\infty) : D^k f(m) = 0 \text{ for every } k \in \mathbb{N} \cup \{0\}\}.$$

Theorem 2.2 can now be reformulated:

COROLLARY 2.4. *Let I be a closed ideal in H^∞ satisfying $Z(I) \subseteq G$. Let*

$$E_n(I) = \{x \in M(H^\infty) : \operatorname{ord}(I, x) \geqslant n\} \qquad (n \in \mathbb{N})$$

be the higher order zero sets (or hulls) of I. Then there exists $N \in \mathbb{N}$ such that $E_n(I) = \varnothing$ for $n \geqslant N+1$ and

$$I = \{f \in H^\infty : D^{(j-1)} f = 0 \text{ on } E_j(I) \text{ for all } j = 1, 2, ..., N\}.$$

In particular, I is uniquely determined by its higher order hulls.

Proof. This follows from Lemma 1.3, Theorem 2.2 and Lemma 2.3. ∎

Remark. We also deduce that I has the following representation:

$$I = I[E_1(I), ..., E_N(I)].$$

3. GENERATORS FOR IDEALS WITH HULL CONTAINED IN G

It is well known (see [5, 17]) that any ideal in H^∞ whose hull does not meet the set of trivial points, is generated algebraically by Carleson–Newman Blaschke products (we already used this fact in the proof of Theorem 2.2).

What we are interested here, is to give precise information on how one may choose these generators. This amounts in a detailed study of the higher order hulls.

First we need some additional results on higher order derivatives. The first result is from Tolokonnikov [17]. For the readers convenience, we would like to present a short proof.

PROPOSITION 3.1 [17]. *Let $f \in H^\infty$ and $n \in \mathbb{N}$. Then $\sum_{k=0}^n |D^k f|$ is bounded away from zero on $\mathbb{D}$ if and only if $f = BF$, where B is a Carleson–Newman Blaschke product of order $p \leqslant n$ and F is invertible in H^∞.*

Proof. Assume that $\sum_{k=0}^n |D^k f| \geqslant \delta > 0$ on $\mathbb{D}$. By Hoffman's theory and the Corona Theorem, this holds on $M(H^\infty)$, too. Since the derivatives $D^j f$ vanish identically for $j = 1, 2, ...$ on the set of trivial points (see [1, 9]), our hypothesis implies that $|f| \geqslant \delta > 0$ on Γ. By [8] $f = BF$ for some Carleson–Newman Blaschke product B and some F invertible in H^∞. Assume that for some m we have $\operatorname{ord}(B, m) \geqslant n+1$. Then $\operatorname{ord}(f, m) \geqslant n+1$ and by Lemma 2.3 we deduce that $D^j f(m) = 0$ for all $j = 0, 1, ..., n$. This is a contradiction.

The converse easily follows from Lemma 2.3 and the fact that a Carleson–Newman Blaschke product B has order N if and only if $E_j(B) = \varnothing$ for $j > N$, but $E_N(B) \neq \varnothing$. ∎

PROPOSITION 3.2. *If $(f_n)_{n \in \mathbb{N}}$ is a sequence in H^∞ converging uniformly on $\mathbb{D}$ to some $f \in H^\infty$, then $(D^j f_n)_{n \in \mathbb{N}}$ converges on $M(H^\infty)$ uniformly to $D^j f$ for every $j \in \mathbb{N}$.*

Proof. It suffices to show that if $f_n \in H^\infty$ tends uniformly to zero, then $D^j f_n$ tends uniformly to zero. Note that

$$D^j f(z) = \sum_{k=1}^{j} a_k \bar{z}^{j-k} (1-|z|^2)^k f^{(k)}(z) \tag{11}$$

for some constants a_k. Hence it is enough to show that $(1-|z|^2)^k f_n^{(k)}(z)$ tends uniformly to zero on $\mathbb{D}$ as $n \to \infty$. But for $0 < r < 1$ and $\|f_n\|_\infty \leqslant \varepsilon$, we have

$$\begin{aligned}
\frac{1}{k!} r^k (1-|z|^2)^k \, |f_n^{(k)}(rz)| &= \left| \frac{(1-|z|^2)^k}{2\pi i} \int_{|\eta|=1} \frac{f_n(\eta r)}{(\eta - z)^{k+1}} \, d\eta \right| \\
&\leqslant \frac{\varepsilon}{2\pi} \int_{|\eta|=1} \frac{(1-|z|^2)^k}{|\eta - z|^{k+1}} \, |d\eta| \\
&\leqslant \frac{\varepsilon}{2\pi} \int_0^{2\pi} \frac{(1-|z|^2)}{|e^{i\theta} - z|^2} \frac{(1-|z|^2)^{k-1}}{(1-|z|)^{k-1}} \, d\theta \\
&\leqslant 2^{k-1} \varepsilon \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta} - z|^2} \, d\theta = 2^{k-1} \varepsilon.
\end{aligned}$$

This yields the assertion. ∎

PROPOSITION 3.3. *Let U_j be open sets in $M(H^\infty)$. Suppose that for some interpolating Blaschke products b_k the Blaschke product $B = \prod_{k=1}^n b_k$ satisfies*

$$E_j(B) \subseteq U_j (j = 1, ..., n). \tag{12}$$

Then there exists $\lambda_0 > 0$ such that for every $f \in H^\infty$, $\|f\|_\infty \leqslant 1$ and $0 < \lambda \leqslant \lambda_0$ the functions $f_\lambda = B + \lambda f$ have the property that

$$E_j(f_\lambda) \subseteq U_j \qquad (j = 1, ..., n). \tag{13}$$

Moreover, the inner factor of f_λ *is a Carleson–Newman Blaschke product of order* $p \leqslant n$ *and the outer factor of* f_λ *is invertible.*

Proof. By Proposition 3.2, $D^j f_\lambda$ converges uniformly to $D^j B$ on $M(H^\infty)$ as $\lambda \to 0$. The continuity of the $D^j f$ and the fact that

$$E_j(f_\lambda) = \left\{ m \in M(H^\infty) : \sum_{k=0}^{j-1} |D^k f_\lambda(m)| = 0 \right\} \qquad (j = 1, 2, ...)$$

(see Lemma 2.3) yields assertion (13).

To prove the remaining assertions, we note that B is a Carleson–Newman Blaschke product of order less than or equal to n. Hence Proposition 3.1 yields that $\sum_{k=0}^{n} |D^k B| \geqslant \delta$ on $\mathbb{D}$. By the uniform convergence of the $D^k f_\lambda$, we see that there exists a positive λ_0 such that

$$\sum_{k=0}^{n} |D^k f_\lambda| \geqslant \frac{\delta}{2}$$

on $\mathbb{D}$ for $0 < \lambda \leqslant \lambda_0$. Note that λ_0 is independent of f. By Proposition 3.1 we obtain that the inner factors of f_λ $(0 < \lambda \leqslant \lambda_0)$ are Carleson–Newman Blaschke products of order less or equal to n and that the outer factors are invertible. ∎

THEOREM 3.4. *Let* I *be an ideal in* H^∞ *such that*

$$N = \sup\{\operatorname{ord}(I, x) : x \in Z(I)\} < \infty. \tag{14}$$

Let U_j *be open sets satisfying* $E_j(I) \subseteq U_j$ $(j = 1, ..., N)$. *Then* I *is algebraically generated by Carleson–Newman Blaschke products* B *of order* N *such that*

$$E_j(B) \subseteq U_j \qquad (j = 1, ..., N).$$

Proof. Let $\Omega_1 = U_1$, $\Omega_j = U_j \setminus \Omega_{j-1}^c$, $j = 2, ..., N$. Then $E_j(I) \subseteq \Omega_j \subseteq U_j$, and $\Omega_{j+1} \subseteq \Omega_j$. We first show that I contains a Carleson–Newman Blaschke product C of order N such that $E_j(C) \subseteq \Omega_j$ $(j = 1, ..., N)$.

To prove this, let $x \in Z(I)$. Choose $p \in \{1, ..., N\}$ maximal, such that $x \in E_p(I)$. Take an open neighborhood U_x of x in $M(H^\infty)$ such that $U_x \subseteq \Omega_p$, but $U_x \cap E_{p+1}(I) = \emptyset$.

Choose $f_x \in I$ satisfying $\operatorname{ord}(f_x, x) = \operatorname{ord}(I, x)$. Let $s = s(x) = \operatorname{ord}(f_x, x)$. By Hoffman's factorization theorem ([9], p. 100), $f_x = c_{x,1} c_{x,2} \cdots c_{x,s}\, g_x$, where $c_{x,j}$ is an interpolating Blaschke product with $c_{x,j}(x) = 0$ and $g_x \in H^\infty$, $g_x(x) \neq 0$.

Since $Z(I)$ is totally disconnected, there exist open sets V_x in $M(H^\infty)$ such that

$$x \in V_x \subseteq \overline{V_x} \subseteq U_x, \qquad V_x \cap Z(g_x) = \emptyset,$$

and $V_x \cap Z(I)$ is a clopen subset of $Z(I)$.

Since $Z(I)$ is compact, there exist $x_1, x_2, ..., x_n \in Z(I)$ such that

$$Z(I) \subseteq \bigcup_{j=1}^{n} V_{x_j} \qquad \text{and} \qquad V_{x_j} \cap Z(I) \nsubseteq \bigcup_{i:\, i \neq j} (V_{x_i} \cap Z(I)).$$

Let $A_1 = V_{x_1} \cap Z(I)$ and $A_j = (V_{x_j} \cap Z(I)) \setminus \bigcup_{i=1}^{j-1} (V_{x_i} \cap Z(I))$, $j = 2, ..., n$. Then the A_j are pairwise disjoint, nonvoid clopen subsets of $Z(I)$ such that $\bigcup_{j=1}^{n} A_j = Z(I)$ and $A_j \subseteq V_{x_j} \cap Z(I)$.

Take an open subset W_j in $M(H^\infty)$ such that

$$A_j \subseteq W_j \subseteq \overline{W_j} \subseteq V_{x_j}, \qquad \overline{W_j} \cap \overline{W_i} = \emptyset \quad \text{if } i \neq j, \qquad A_j = W_j \cap Z(I).$$

Then we see that $W_j \cap Z(I) = \overline{W_j} \cap Z(I)$ and

$$Z(I) \subseteq \bigcup_{j=1}^{n} W_j. \tag{15}$$

We note that x_j does not necessarily belong to W_j.

Let $C_{x_j, i}$ be the subproduct of $c_{x_j, i}$ with $Z(C_{x_j, i}) \cap \mathbb{D} = Z(c_{x_j, i}) \cap \mathbb{D} \cap W_j$. Then $Z(C_{x_j, i}) \subseteq \overline{W_j}$. Moreover,

$$f_{x_j} = C_j h_j, \qquad \text{where} \quad C_j = \prod_{i=1}^{s(x_j)} C_{x_j, i} \qquad \text{and} \qquad h_j \in H^\infty, \qquad (j = 1, ..., n).$$

Since $\bigcap_{j=1}^{n} Z(h_j) \cap Z(I) \overset{(15)}{=} \emptyset$, we obtain from Lemma 1.1 that $C := \prod_{j=1}^{n} C_j \in I$. We claim that

$$E_p(C) \subseteq \Omega_p \qquad (p = 1, ..., N). \tag{16}$$

Let $x \in E_p(C)$. Since $x \in \bigcup_{j=1}^{n} Z(C_j)$ and $Z(C_j) \subseteq W_j$, the pairwise disjointness of the W_j implies that there exists a unique $j \in \{1, ..., n\}$ so that $x \in Z(C_j)$. Hence

$$x \in Z(C_j) \subseteq W_j \subseteq V_{x_j} \subseteq U_{x_j} \subseteq \Omega_\ell,$$

where ℓ is chosen so that $x_j \in E_\ell(I) \cap \Omega_\ell$ and so that ℓ is maximal. Note that $x_j \in Z(I)$. Moreover, since the $C_{x_j, i}$ are interpolating Blaschke products, $C_j = \prod_{i=1}^{s(x_j)} C_{x_j, i}$ implies that $p = \operatorname{ord}(C, x) \leqslant s(x_j) = \ell$. Therefore $\Omega_\ell \subseteq \Omega_p$. Hence $x \in \Omega_p$. This shows that $E_p(C) \subseteq \Omega_p$.

By (14), $\sup\{\operatorname{ord}(I, x): x \in Z(I)\} = N$. Thus we see that $s(x_j) \leqslant N$ for every $j \in \{1, ..., n\}$. Hence the C_j are Carleson–Newman Blaschke products of order at most N. Since the $Z(C_j)$ are pairwise disjoint and $C \in I$, by (14) we get that C is a Carleson–Newman Blaschke product of order N.

It remains to show that I is generated by Carleson–Newman Blaschke products B of order N satisfying $E_j(B) \subseteq U_j$ for every $j \in \{1, ..., N\}$. Let $f \in I$, $\|f\|_\infty \leqslant 1$. Taking λ_0 as in Proposition 3.3, we can conclude that the outer factors of the functions $C + \lambda_0 f$ are invertible and that the inner factors of them are just the Carleson–Newman Blaschke products B of order N satisfying $E_j(B) \subseteq U_j$ for every $j \in \{1, ..., N\}$ we have been looking for. The collection of these B's together with C is now our generating set for the ideal I. ∎

4. APPLICATIONS TO FINITELY GENERATED IDEALS AND FINITE PRODUCTS OF IDEALS

Let $f_1, ..., f_n \in H^\infty$ and let

$$I = I(f_1, ..., f_n) = \left\{ \sum_{j=1}^{n} g_j f_j : g_j \in H^\infty \right\}$$

be the ideal generated by the functions f_j. Let

$$\begin{aligned} J &= J(f_1, ..., f_n) \\ &= \left\{ f \in H^\infty : |f| \leqslant C \sum_{j=1}^{n} |f_j| \text{ on } \mathbb{D} \text{ for some } C = C(f) \right\}. \end{aligned}$$

It is well-known that $I \subseteq J$ and that, in general, the inclusion is strict (see [4, p. 369]). T. Wolff showed that $f \in J$ implies that $f^3 \in I$ (see [4, p. 329]). It is not known whether $f \in J$ implies $f^2 \in I$. In [7], however, it is shown that if the hull of I is contained in the set G of nontrivial points, then $f \in J$ implies that $f^2 \in I$. J. Bourgain [2] proved that $f \in J$ implies $f^2 \in \overline{I}$, the closure of I, for any finitely generated ideal I in H^∞. As a corollary of Theorem 2.2 and Corollary 2.4 we now obtain:

THEOREM 4.1. *Let $I = I(f_1, ..., f_n)$ be a finitely generated ideal in H^∞ such that $Z(I) \subseteq G$. Then $f \in J$ implies that $f \in \overline{I}$.*

Proof. Just note that $E_k(I) = E_k(J)$ for every $k \in \mathbb{N}$ and use Corollary 2.4. ∎

Remarks. (1) Bourgain gave an example of two Blaschke products B and C such that $BC \notin \overline{I(B^2, C^2)}$, although $|BC| \leqslant |B^2| + |C^2|$. Thus the condition "$Z(I) \subseteq G$" cannot be removed to conclude that "$f \in J$ implies $f \in \bar{I}$". (See [6] for a systematic approach to Bourgain's example).

(2) Theorem 2.2 in [15], which is a special case of Theorem 2.2 here, was proven by entirely different methods and applies only to powers of two interpolating Blaschke products. In addition, we obtain the following result mentioned in [15] as Remark 2:

Let B and C be two interpolating Blaschke products. Then

$$\begin{aligned} J(B^N, C^N) &\subseteq \overline{I(B^N, C^N)} \\ &= \{ f \in H^\infty : \operatorname{ord}(f, m) \geqslant N \text{ for all } m \in Z(B) \cap Z(C) \}. \end{aligned}$$

In the remainder of this section we consider products of ideals. The setting is the following: For $j = 1, ..., n$ let I_j be ideals in H^∞. The tensor product $\bigotimes_{j=1}^n I_j = I_1 \otimes \cdots \otimes I_n$ is defined to be the set of all finite sums $\sum_{k=1}^K f_{k,1} \cdots f_{k,n}$, where the $f_{k,j} \in I_j$ for $j = 1, ..., n$. These products play an important role in describing the ideals $J(f_1, ..., f_n)$. For example we have that $J(B^N, C^N) = \bigotimes_{j=1}^N I(B, C)$ whenever B and C are interpolating Blaschke products (see [15]). In order to prove our next result, we begin with a result of Lingenberg. First recall that a set $E \subseteq M(H^\infty)$ is ρ-separated if $\rho(x, y) \geqslant \delta > 0$ for every $x, y \in E$, $x \neq y$.

LEMMA 4.2 [13, pp. 59–60]. *Let B be a Carleson–Newman Blaschke product and let E be a closed ρ-separated subset of $Z(B)$. Then there exists an interpolating factor b of B, such that $E \subseteq Z(b)$.*

THEOREM 4.3. *Let I be a closed ideal in H^∞ satisfying*

$$(1) \quad Z(I) \subseteq G \qquad \textit{and} \qquad (2) \quad Z(I) \textit{ is } \rho\textit{-separated.}$$

Let

$$N = \sup\{ \operatorname{ord}(I, x) : x \in Z(I) \}.$$

Then $N < \infty$ and

$$I = I[E_1(I), ..., E_N(I)] = \bigotimes_{j=1}^{N} I[E_j(I)].$$

Moreover the ideals $I[E_j(I)]$ are generated by interpolating Blaschke products.

Proof. By Theorem 2.2 $I = I[E_1(I), ..., E_N(I)]$. Let $f \in I$ be a Carleson–Newman Blaschke product. Since $E_1(I) = Z(I)$ is ρ-separated, by Lemma

4.2 there exists an interpolating Blaschke product b_1 with $Z(b_1) \supseteq E_1(I)$ such that $f = b_1 g_1$ for some Carleson–Newman Blaschke product g_1. Note that $\operatorname{ord}(b_1, x) = 1$ for every $x \in Z(b_1)$. Since $f \in I$, we see that $E_2(f) \supseteq E_2(I)$. Hence $g_1(x) = 0$ for every $x \in E_2(I)$. Moreover $E_2(I)$ is a ρ-separated subset of $Z(g_1)$. Again, by Lemma 4.2, there exists an interpolating Blaschke product b_2 such that $g_1 = b_2 g_2$ with $Z(b_2) \supseteq E_2(I)$ for some Carleson–Newman Blaschke product b_2. After N-steps, we obtain N interpolating Blaschke products b_j satisfying $Z(b_j) \supseteq E_j(I)$, so that $f = b_1 b_2 \cdots b_N g$ for some $g \in H^\infty$. Hence $f \in \prod_{j=1}^N I[E_j(I)]$. Since I is generated by Carleson–Newman Blaschke products we see that I is contained in the ideal generated by the set $\prod_{j=1}^N I[E_j(I)]$. Thus $I \subseteq \bigotimes_{j=1}^N I[E_j(I)]$.

Conversely, if $f \in \bigotimes_{j=1}^N I[E_j(I)]$, then $f = \sum_{k=1}^p \prod_{j=1}^N f_{k,j}$ for some $f_{k,j} \in I[E_j(I)]$. Since $E_1(f) \supseteq E_2(f) \supseteq \cdots \supseteq E_N(f)$, it immediately follows that for $x \in E_\ell(I)$ we have $\operatorname{ord}(f, x) \geqslant \ell$. Hence $f \in I[E_1(I), ..., E_N(I)]$. We conclude that

$$I[E_1(I), ..., E_N(I)] = \bigotimes_{j=1}^N I[E_j(I)].$$

To prove the assertion about the generators, we note that $E_j(I)$ is a closed ρ-separated set contained in G. Moreover, $Z(I[E_j(I)]) \subseteq Z(I) \subseteq G$. Hence, by [13, Corollary 3.5 and Theorem 3.3] or [10, p. 552], we have $E_j(I) \subseteq Z(b)$ for some interpolating Blaschke product b. By [14], $I[E_j(I)]$ then is generated by interpolating Blaschke products. ∎

Remark. In order to obtain the product representation of Theorem 4.3 we note that the assumption that $Z(I)$ be ρ-separated is necessary. To see this, let B and C be two interpolating Blaschke products such that $Z(B) \cup Z(C)$ is not ρ-separated. Let $I = BCH^\infty$. Then, trivially, I is closed with $E_1(I) = Z(B) \cup Z(C)$ and $E_2(I) = Z(B) \cap Z(C)$. Moreover $E_3(I) = \varnothing$. But as we show below, the ideal $I^* = I[E_1(I)] \otimes I[E_2(I)]$ satisfies $E_3(I^*) \neq \varnothing$.

In fact, let $x_n \in Z(B)$, $y_n \in Z(C)$ so that $\rho(x_n, y_n) \to 0$, $x_n \neq y_k$ for all $n, k \in \mathbb{N}$. Let $f \in I[E_1(I)]$. Then $f(x_n) = f(y_n) = 0$. Hence, by ([11], p. 442), we have $\operatorname{ord}(f, x) \geqslant 2$ for every cluster point x of $\{x_n : n \in \mathbb{N}\}$. By the lower semicontinuity of the pseudohyperbolic distance ρ, the point x also is a cluster point of the y_n. Therefore $x \in E_2(I)$ and so $x \in E_3(I^*)$.

We conclude this paper with the following open problems:

(1) Let I be an ideal in H^∞. Is $\prod_{j=1}^N I[E_j(I)]$ an ideal? In other words, do we have

$$\prod_{j=1}^N I[E_j(I)] = \bigotimes_{j=1}^N I[E_j(I)]?$$

By Hoffman's theory, we know that this is true if the $E_j(I)$ are singletons.

(2) Under the assumptions of Theorem 4.3 the ideal $\bigotimes_{j=1}^{N} I[E_j(I)]$ is closed. Now let E_j be closed ρ-separated subsets of G, $(j=1, ..., n)$. Is the product $\bigotimes_{j=1}^{n} I[E_j]$ then a closed ideal? If this were true, then

$$\bigotimes_{j=1}^{n} I[E_j] = I[S_1, ..., S_n],$$

where

$$S_1 = \bigcup_{1 \leqslant j \leqslant n} E_j,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$S_p = \bigcup_{1 \leqslant j_1 < \cdots < j_p \leqslant n} (E_{j_1} \cap \cdots \cap E_{j_p}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$S_n = \bigcap_{1 \leqslant j \leqslant n} E_j,$$

are the higher order hulls.

In the special case that $E_j = Z(B_j) \cap Z(C_j)$ $(j=1, 2)$ for interpolating Blaschke products B_j and C_j, the question has an affirmative answer. In fact there we have:

$$\begin{aligned}\overline{I(B_1 B_2, B_1 C_2, C_1 B_2, C_1 C_2)} &= \overline{I(B_1, C_1) \otimes I(B_2, C_2)} \\ &= I[E_1] \otimes I[E_2] = I[E_1 \cup E_2, E_1 \cap E_2].\end{aligned}$$

The most important open problem in this context, however, is the following:

(3) Give a characterization of the higher order hulls $E_j(I)$ of ideals in H^∞.

REFERENCES

1. S. Axler and K. Zhu, Boundary behaviour of derivatives of analytic functions, *Michigan Math. J.* **39** (1992), 129–143.
2. J. Bourgain, On finitely generated closed ideals in $H^\infty(\mathbb{D})$, *Ann. Inst. Fourier (Grenoble)* **35** (1985), 163–174.
3. P. Budde, Support sets and Gleason parts, *Michigan Math. J.* **37** (1990), 367–383.
4. J. B. Garnett, "Bounded Analytic Functions," Academic Press, New York, 1981.
5. P. Gorkin and R. Mortini, Interpolating Blaschke products and factorization in Douglas algebras, *Michigan J. Math.* **38** (1991), 147–160.

6. P. Gorkin and R. Mortini, A survey of closed ideals in familiar function algebras, *in* "Proc. of the 3rd Conference on Function Spaces, SIUE, Edwardsville" (K. Jarosz, Ed.), *Contemp. Math.* **232** (1999), 161–170.
7. P. Gorkin, R. Mortini, and A. Nicolau, The generalized corona theorem, *Math. Annal.* **301** (1995), 135–154.
8. C. Guillory, K. Izuchi, and D. Sarason, Interpolating Blaschke products and division in Douglas algebras, *Proc. Roy. Irish Acad. Sect. A* **84** (1984), 1–7.
9. K. Hoffman, Bounded analytic functions and Gleason parts, *Ann. of Math.* **86** (1967), 74–111.
10. K. Izuchi, Interpolating Blaschke product s and factorization theorems, *J. London Math. Soc.* **50** (1994), 547–567.
11. K. Izuchi and Y. Izuchi, Inner functions and division in Douglas algebras, *Michigan Math. J.* **33** (1986), 435–443.
12. B. Korenblum, Closed ideals in the ring A^+, *Funct. Anal. Appl.* **6** (1972), 203–214.
13. H.-M. Lingenberg, Interpolating sets in the maximal ideal space of H^∞, *Michigan Math. J.* **39** (1992), 53–63.
14. R. Mortini, Ideals generated by interpolating Blaschke products, *Analysis* **14** (1994), 67–74.
15. R. Mortini, On an example of J. Bourgain on closures of finitely generated ideals in H^∞, *Math. Z.* **224** (1997), 655–663.
16. V. Tolokonnikov, Interpolating Blaschke products and ideals of the algebra H^∞, *Soviet J. Math.* **27** (1984), 2549–2553.
17. V. Tolokonnikov, Blaschke products satisfying the Carleson–Newman condition and ideals of the algebra H^∞, *Soviet J. Math.* **42** (1988), 1603–1610.

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